

Order and supersymmetry at high filling zero-energy states on the triangular lattice

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We perform exact diagonalization studies in $d = 2$ dimensions for the Fendley and Schoutens model of hard-core and nearest-neighbor excluding fermions that displays an exact non-relativistic supersymmetry. Using clusters of all possible shapes up to 46 sites, we systematically study the behavior of the ground state phase diagram as a function of filling. We focus on the highly degenerate zero-energy states found at fillings between $1/7$ and $\sim 1/5$. At the lower end of that interval, at filling $1/7$, we explicitly show that the ground states are gapped crystals. Consistent with previous suggestions, we find that the extensive entropy of zero states peaks at a filling of ~ 0.178 . At the higher end of the interval, we find zero energy ground states at fillings above $1/5$, in contradiction to previous numerical studies and analytical suggestions; these display non-trivial amplitude degeneracies.

I. INTRODUCTION

Supersymmetry is an extended symmetry which partners together fermions and bosons and is typically studied in the context of extensions of the high-energy Standard model of particle physics¹. Recently, Fendley and Schoutens^{2,3} proposed a many-body lattice model which exhibits what is known as supersymmetric quantum mechanics^{4,5}. It consists of hard-core lattice fermions with nearest neighbor exclusion and a precisely fine-tuned interaction coupling. The eigenstates of the model consist of positive doubly degenerate energy states where the supersymmetry is spontaneously “broken” and unpaired zero-energy ground states. The nature of these states was studied extensively on the square lattice⁶, in specially decorated two-dimensional lattices^{7,8}, and also in one-dimensional models, where it provided insights into hidden many-body symmetries⁹.

On the square lattice the number of zero-energy states was exponential only in the system’s *linear* dimension⁶, leading to a subextensive entropy of possible ground-states. In contrast, on the triangular lattice the number of zero energy states appears exponential in the *area*, as recent exact diagonalization (ED) studies have indicated¹⁰. Jonsson, by studying finite triangular clusters, recently conjectured¹¹, that on the triangular lattice, zero-energy states only appear in the interval $1/7 \leq f \leq 1/5$, where f is the filling (number of fermions/site).²⁸ Huijse *et al.*¹⁰ recently investigated, using ED, the nature and number of zero-energy states on the triangular lattice. Their results appear consistent with Jonsson’s conjecture and the suggestion of exponential zero energy states’ degeneracy, but they focused on quasi-one dimensional ladders, of width up to four.

In this paper, we use exact diagonalization applied to *all* symmetrically inequivalent periodic clusters of various shapes and sizes, to determine the phase diagram as a function of filling. By combining data from many dif-

ferent clusters, we are able to obtain curves of the energy as a function of filling across the whole phase diagram, and the entropy of zero states as a function of filling, across the interval in which they are found. Our study emphasizes *two*-dimensional clusters, wider than those of Ref.¹⁰.

The outline of our paper and main findings are as follows. First, in Section II, we introduce the supersymmetric lattice model and the parameters of our exact diagonalization calculations. In Section III we present our results on the different phases that appear as the density of fermions increases: a Fermi liquid phase at low filling $f < 1/7$, zero-energy states with extensive entropy at intermediate fillings, and high-density states (with tendencies to spatial orders) from $f \approx 1/5$ up to the maximum filling $f = 1/3$. In Section IV, we focus on the states found near either endpoint of the zero-energy ground state interval. The minimum filling with a zero energy state is exactly $f = 1/7$ as conjectured¹¹; however, from that filling up to $f \approx 0.156$, the only zero-energy states are crystal-like and gapped states, and have no extensive entropy. At fillings slightly higher than $1/5$, we find that certain zero-energy states exist, contradicting the conjecture¹¹. These states show a tendency to anisotropic forms of spatial order, and their wavefunctions exhibit surprising regularities, in that many inequivalent fermion configurations have the same, maximal amplitude in the wavefunction (Sec. IV B 3). Finally, Section V contains a summary of our results and speculations on how these results may fit into a complete picture of the phase diagram.

II. MODEL AND METHOD

The Hamiltonian of the model is based on the definition of the operators $Q = \sum_i c_i^\dagger P_{\langle i \rangle}$ and $Q^\dagger = \sum_i c_i P_{\langle i \rangle}$, where $P_{\langle i \rangle} = \prod_{j \text{ next to } i} (1 - c_j^\dagger c_j)$ form projectors that

exclude nearest neighbor occupancy. Following the basic recipe on constructing the supersymmetric quantum mechanics¹², the Hamiltonian is just

$$H = \{Q^\dagger, Q\} = t \sum_i \sum_{j \text{ next to } i} P_{\langle i \rangle} c_i^\dagger c_j P_{\langle j \rangle} + V \sum_i P_{\langle i \rangle}, \quad (1)$$

with $t \equiv V$. The first term in (1) is a kinetic term of nearest-neighbor hopping (subject to the hard-core constraint); the second term is a potential energy, equal to the number of fermions plus the number of vacant sites *not* forbidden by a neighboring fermion, which effectively includes pairwise farther-neighbor interactions as well as multi-fermion interactions. The Hamiltonian (1) is supersymmetric only when finely tuned to the special point $t = V$. In view of the massive ground-state degeneracy, any change in the Hamiltonian would be a singular perturbation leading to a new phase, if the energy spectrum is gapless; hence, the $t = V$ point in the extended phase diagram is speculated to be a multicritical point at which various phases meet⁶.

The model contains an exact $\mathcal{N} = 2$ supersymmetry in that all eigenstates with nonzero energy E belong to degenerate *pairs*, $|a\rangle$ with F fermions and $|b\rangle$ with $F + 1$ fermions, such that $Q|a\rangle = \sqrt{E}|b\rangle$ and $Q^\dagger|b\rangle = \sqrt{E}|a\rangle$, whereas $Q|b\rangle = Q^\dagger|a\rangle = 0$. The state with an even number is considered “bosonic” and the one with an odd number is considered “fermionic”. An interesting corollary is that if there is a branch of fermionic elementary excitations, there must be a corresponding bosonic branch and vice versa.

Most importantly, there may be ground states $|G\rangle$ of zero energy, such that $Q|G\rangle = Q^\dagger|G\rangle = 0$. A thermodynamically *extensive* zero-energy ground state entropy was reported and analyzed in several lattices, including integrable chains or ladders^{6,7} as well as two-dimensional lattices, of which the triangle lattice is the simplest. In each model, the zero-energy states are always limited to fillings in the interval $f \in (f_-, f_+)$, where f is the number of fermions per site. These prior studies mostly indicated that the zero-energy states have a strong tendency to form crystalline phases.

We diagonalize the model of Eq. (1) on a triangular lattice using clusters of different size and shapes. Each cluster is characterized by the shortest and the next to shortest edge, $\mathbf{v} = (v_1, v_2)$ and $\mathbf{u} = (u_1, u_2)$ respectively, where the integer coordinates $u_{1,2}$ and $v_{1,2}$ are expressed relative to the triangular lattice basis vectors $\hat{e}_1 = \hat{x}$ and $\hat{e}_2 = -\frac{1}{2}\hat{x} + \frac{\sqrt{3}}{2}\hat{y}$; the number of sites is $N = u_1v_2 - u_2v_1$. We used *all* the symmetrically inequivalent clusters such that \mathbf{v} belongs to the irreducible wedge of the triangular lattice and $|\mathbf{v}|^2 \leq |\mathbf{u}|^2$ for up to $N = 46$ sites (larger for some fillings). We perform the diagonalization separately for each sector defined by fermion count F and center of mass momentum. For the diagonalization we use the ARPACK package¹³, which is an implementation of the Implicitly Restarted Lanczos Method (IRAM). Note that in accordance to Ref. 10, well within the zero state in-

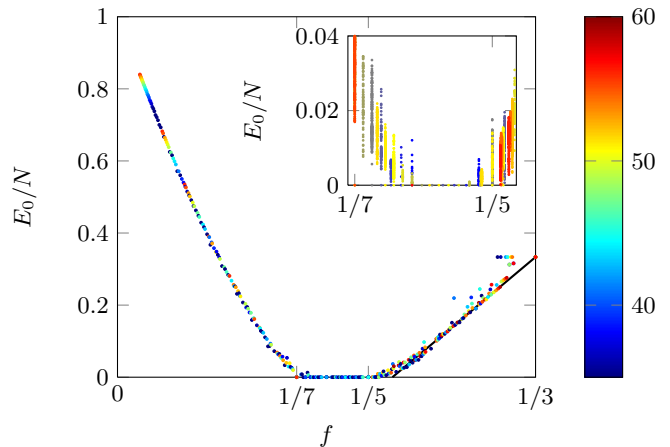


FIG. 1: Ground state energy per site vs. filling, minimized over all possible clusters with sizes $35 \leq N \leq N_{\max}(f)$, where $N_{\max}(f) = 56$ for $f < 1/7$, 46 for $1/7 \leq f \leq 1/5$ and 60 for $f > 1/5$. To filter out ladders we also impose the constraint $|\mathbf{v}| \geq 4$. For each filling the energy is minimized across all clusters and momenta. The different data points are colored according to the total system size. The solid line is a linear low envelop of the high energy regime which has the form $E = \frac{1}{3} + \lambda(f - \frac{1}{3})$, with $\lambda = 2.92$. In the inset we show the energy versus filling for all the clusters zooming in the interval of zero energy states.

terval we find that every momentum sector has a similar number of zero energy states, about 100 per sector in our largest system ($N = 46$); it is this huge degeneracy that limited the cluster size we could handle in this range of fillings. For other fillings, namely $f < 1/7$ and $f > 1/5$ we study up to $N = 56$ and $N = 60$ respectively. In any case, for comparison, the tractable clusters are typically much larger than in Hubbard model exact diagonalization studies, since there is only one spin species and a hardcore constraint is enforced, each of which greatly reduces the Hilbert space.

III. RESULTS: PHASE DIAGRAM

The model of Eq. (1) displays three clear regimes as a function of particle density as evidenced from the equation of state in Fig. 1 and in the configurations illustrated in Fig. 2. The qualitative behavior at low and high density follows the studies by Henley and Zhang^{14,15}, who previously used exact diagonalization to study a model which was similar in that it has the same Hilbert space – spinless fermions with nearest-neighbor exclusion – but having $V = 0$ in Eq. (1), so it lacked supersymmetry. At low densities a Fermi liquid phase appears, followed by zero energy ground states for fillings $1/7 < f < 0.2174$. At the maximum allowed density of $f = 1/3$, the only allowed configuration is a crystalline phase with no quantum fluctuations, which, dominates the phase diagram beyond the interval of zero-energy states. In the rest of

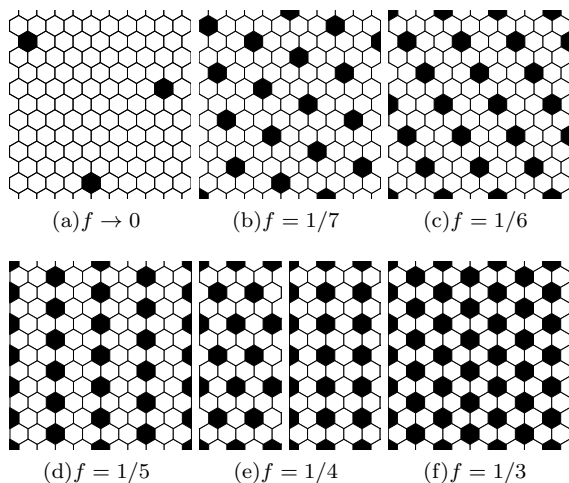


FIG. 2: **Idealized fermion configurations representing phases at various fillings.** In this and subsequent figures, the triangular lattice sites are the centers of the hexagons, and filled hexagons are spinless fermions. For each filling, a configuration of minimal potential energy is shown. More generally, depending on the unit cell constraints of each particular cluster, the configurations of highest weight and having minimum potential energy tend to be mixtures of triangles from the crystals shown here. (a) At low density the system is a dilute gas of weakly interacting fermions. (b) A triangular superlattice crystal at $f = 1/7$ minimizes the potential energy for that filling. (c) A covering configuration at $f = 1/6$ (d) Two possible covering configurations for $f = 1/4$ (e) Anisotropic triangular lattice at $f = 1/5$ (f) At $f = 1/3$, there are only three allowed configurations in which one of 3 sublattices is fully occupied and no hopping is possible.

this section we discuss those three different regimes of the phase diagram.

A. Fermi liquid phase ($f < 1/7$)

The low density Fermi liquid is similar to the one studied by Henley and Zhang^{14,15} in their non-supersymmetric model. The ground state for each filling, at low density, has the quantum numbers and is approximated by the corresponding ground state for non-interacting fermions on the same cluster. A consequence of the positive hopping $t > 0$ in our Hamiltonian is that the single fermion dispersion has *two* symmetry-related valleys, with minima at the Brillouin zone corners. For a given cluster, the sequence of energies as fermions are added one by one shows “shell filling effects” as expected in a Fermi liquid¹⁶: that is, in the dilute limit two successive fermions have a similar addition cost (placing one in each valley), then the next two are higher. Thus the addition energies show an even/odd alternation on top of an overall increasing trend, due to the repulsive (hard-core) interaction. If it were necessary to incorporate interaction effects in a systematic way, one relatively simple

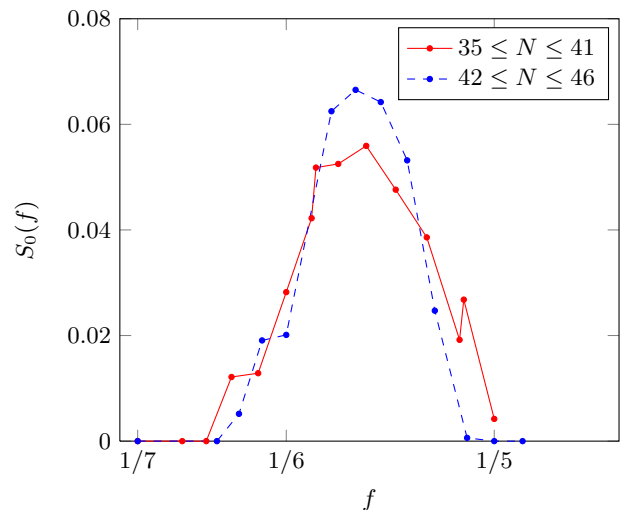


FIG. 3: **Zero energy states entropy.** The zero state entropy per site, averaged over all the clusters with the same filling f for different subsets of clusters. We imposed the constraint $|\mathbf{v}| \geq 4$ to filter out the ladders which have a zero energy state at lower fillings. The solid line corresponds to all cluster with $N \geq 34$ the solid green to clusters with $35 \leq N \leq 41$ and the dot dashed line to $35 \leq N \leq 46$. There seems to be a maximum in the vicinity of $f = 0.178$.

approach would be the t -matrix formalism of Ref. 17.

B. Zero-energy states ($1/7 < f < f_+$)

The degeneracy of the zero-energy states is best characterized by their zero state entropy¹⁸, which is extensive in the intermediate fillings. This fact suggests that at the infinite system size limit, the wave function incorporates a mix of local patterns (especially fragments of the crystals in Figure 2. However, at finite size particular periodic boundary conditions may be commensurate with only a subset of these; hence, different clusters may show contrasting behaviors, which would *all* be present in different patches of a typical infinite-size configuration.

Let $A_0(F; \mathcal{C})$ be the number of zero energy states for each given cluster \mathcal{C} ; this is one of the outputs of the ED. We conjecture that $A_0(F; \mathcal{C}) \sim \exp(NS_0(f))$, where $N(\mathcal{C})$ is the number of sites in the cluster and $f = F/N$, with an entropy function $S_0(f)$ that is well-defined in the thermodynamic limit. We construct a numerical approximation to the entropy function by first computing an entropy per site for each cluster and then averaging over all clusters (of any size) with the identical filling per site, that is

$$S_0(f) = \left\langle \frac{1}{N(\mathcal{C})} \log [N_0(fN(\mathcal{C}); \mathcal{C})] \right\rangle_{\mathcal{C}}. \quad (2)$$

This averaging is meant to reduce the commensurability effects of different cluster shapes.

To better approximate the entropy of a two dimensional system we excluded ladder-shaped clusters by putting a lower limit for the width of \mathbf{v} . We found that ladder-like clusters tend to have larger entropies per site than the more two-dimensional ones, and also this entropy tends to peak at a smaller filling f . Including such clusters would have incorrectly biased our estimates and can spuriously suggest that $S_0(f)$ has another peak below $f = 1/6$.

The result is shown in Fig. 3, where we can observe four interesting numerical trends

- (1) There are no zero-energy states below $1/7$, verifying recent expectations¹¹ and in agreement with recent numerical exact diagonalization studies.¹⁰
- (2) In the interval $1/7 < f < 0.156$ almost all clusters either have no zero energy states or have unique zero energy states in isolated momentum sectors. (In some cases there are doubly degenerate zero energy states due to rotational symmetries.) That accounts for the small fluctuations of the entropy in this range, visible in Fig. 3. In addition, as can be seen in the inset of Fig. 1, the zero energy states for $f < 0.156$ are gapped.
- (3) the bulk of the zero energy states lie in the interval $0.156 \lesssim f \lesssim 1/5$. In this interval the majority of the clusters have degenerate zero energy states in every momentum sector. Let (f_0, S_{\max}) be the maximum point of $S_0(f)$ function; our results indicate that $f_0 \approx 0.178$ and $S_{\max} \approx 0.152$. Van Eerten¹⁹, using a transfer matrix, found a numerical bound on the Witten index, which corresponds¹⁰ to $S_{\max} \geq 0.13$, and also indicates $f_0 \approx 0.18$, with which our results are consistent.
- (4) Finally, at the upper end of the interval of zero states in the vicinity of $f \simeq 1/5$, the slope of the entropy function $S_0(f)$ appears to become much steeper. Thus we cannot rule out the possibility that the zero-energy state interval may end below or at $f = 1/5$ in the thermodynamic limit.

We have also made a study of the energy gaps in this filling interval in clusters that have zero states, i.e. the smallest non-zero eigen-energy (from any momentum sector). The main prior result about gaps in this model on the triangular lattice is that ladders with $f = 1/6$ are gapless¹⁰.

As just noted, zero-energy states with $f < 0.156$ appear to be gapped. In the filling interval $(0.156, 1/5)$ we attempted to study the gaps by averaging over different clusters, in the same spirit as Eq. (2). However, although we see clear trends, the behavior of the gaps depends on the clusters in an apparently irregular way, so the results are trustworthy only within an order of magnitude. Since the gaps are strongly (but not always monotonically) decreasing with N , we only average within groups of clusters having the same (F, N) .

Within our two-dimensional clusters having typically ~ 40 sites, there seemed to be particularly low gaps around two fillings $f \gtrsim 0.170$ and $f \approx 0.190$, with a maximum around $f \approx 0.179$. It appears, in fact, that these two minima represent a supersymmetric pair with F and $F + 1$ fermions, e.g. fillings $7/42$ and $8/42$, and we conjecture that in the thermodynamic limit there is one minimum occurring at $f \approx 0.178$, where the entropy function is highest. The gap values at the minimum decay at least as fast as $1/2^F$ (up to $F = 8$ where our data is complete), and are 10^{-3} – 10^{-1} in our largest systems. At other fillings around the middle of the interval $(0.156, 1/5)$ we still see a decaying trend with F , but slower. For $f \lesssim 0.163$ and $f \gtrsim 0.195$ gaps larger than 0.1 are seen even in our largest systems, so we cannot definitely say whether these are gapped.

C. Phase diagram at filling $f \lesssim 1/3$

It is generically expected that strongly interacting lattice fermion models at high densities phase separate into a high-density insulator, such as the inert $f = 1/3$ crystal in Figure 2(f), and a low-density liquid phase in which kinetic energy is dominant. On the triangular lattice, we would guess that the coexisting liquid density falls around $f = 0.21$ where the Hilbert space is largest. (We used the Pauling approximation for the entropy of allowed configurations, following the Appendix of Ref. 14.) That is roughly the filling at which the dense liquid is becoming congested, hence less favorable, due to the hard-core constraint.

Just which dense state does this dense liquid coexist with? Prior research^{14,15} on the non-supersymmetric model on the square lattice showed that the dense phase is *not* the maximally occupied crystal. Instead, it is a crystal containing a dilute array of quantum-fluctuating strings, which we call “stripe-walls” since each has a deficit of fermion density and is a domain wall of the crystal order²⁹. There is no previous literature on such stripe-walls in the case of the triangular lattice for *fermions*, only for hardcore *bosons* (in the context of ^4He adsorbed on carbon nanotubes^{20,21}). Nevertheless, it was shown in Ref. 14 that in an isolated stripe-wall, hardcore constraints do not allow particle exchanges, hence the fermion and boson cases should be in fact equivalent.

The stripe-wall runs perpendicular to one of the bond directions; fermions on the edge of a domain are free to hop in that direction, owing to the relative shift of the other domain. The deficit of electrons is $1/3$ per step along the stripe-wall. Let the energy per step be E_{SW} : then the chemical potential associated with (non-interacting) stripe-walls is $\mu_{\text{SW}} = 3E_{\text{SW}}$. Thus, a phase of dilute stripe walls is represented by an energy function $E(f) = 1/3 - \mu_{\text{SW}}(1/3 - f)$. $E(1/3) = 1/3$ in the maximally filled crystal, since there is no kinetic energy and the potential energy in (1) is $F = N/3$.

Note that as f continues to decrease and the array

becomes less dilute, collisions between adjacent stripe-walls (due to their quantum fluctuations) start to become significant. This will typically reduce the kinetic energy, causing the $E(f)$ curve to bend upwards. Thus, if μ_{SW} exceeds the slope of a coexistence curve connecting to the $f = 1/3$ crystal, the stripe-wall array is stable at $f \lesssim 1/3$. Its stability ends at the point where a tangent to the $E(f)$ curve can be drawn to the coexisting liquid phase. (Such a point should exist since $E(f)$ is upwards curving.) By contrast, if μ_{SW} is less than the slope of the coexistence curve, the $f = 1/3$ crystalline phase coexists directly with a hole-rich phase having $f \approx 0.22$. In the latter case, in the grand canonical-like ensemble, a first-order transition would be seen between the hole-rich phase and the $f = 1/3$ crystal. To check which scenario holds in the supersymmetric model, we must calculate the energy E_{SW} .

Past studies^{15,20} showed that a single stripe-wall can be mapped exactly to a one-dimensional chain with *noninteracting* spinless fermions at half filling, with their hopping amplitude equal to $t = 1$. This mapping remains valid in the supersymmetric case, since all the accessible configurations for the stripe-wall have equal (and maximal) potential energy F . Thus, the kinetic energy per step is $-2/\pi$, or $-6/\pi$ per removed fermion; the potential energy is $+1$ per removed fermion. Hence, we obtain $\mu_{\text{SW}} = (1 + 6/\pi) = 2.91$.

For comparison, our numerics from exact diagonalization of the supersymmetric model showed that in the interval below $f = 1/3$, the energy is linear as a function of filling with a form $E(f) = \frac{1}{3} + \lambda(f - \frac{1}{3})$, with $\lambda = 2.92$, represented by the solid line (cf. Fig. 1). Since we find that $\lambda \cong \mu_{\text{SW}}$, this leaves undecided whether a stripe-wall array is stable or the $f = 1/3$ crystal coexists directly with $f \approx 0.22$.

The straight line fitting $E(f)$ would pass through $f \approx 0.22$, close to the maximum filling of special zero-energy states we found above $f = 0.2$. However, these states do not appear in Figure 1, which is limited to robustly two-dimensional clusters; the numerical $E(f)$ curve curves upwards in the vicinity of $f = 0.22$ and appears to hit the $E = 0$ axis at $f = 0.20$ – 0.205 . That means the zero-energy states have a smooth transition to $E > 0$ states, which are stable in a short interval up to $f = 0.22$, and then possibly coexisting with either a stripe-array or the $f = 1/3$ crystal.

IV. ZERO-ENERGY STATES AT MAXIMUM AND MINIMUM FILLINGS

The entropy of zero states vanishes, according to Figure 3, at filling $1/7$ and slightly over filling $1/5$. Around those fillings, if zero states exist at all, they tend to be unique in their momentum sector and have other special properties, including a tendency to crystal-like spatial orderings.

N	n	\mathbf{u}	\mathbf{v}
28	4	(6, 2)	(1, 5)
28	4	(6, 4)	(2, 6)
28	4	(10, 2)	(1, 3)
35	5	(8, 5)	(1, 5)
35	5	(13, 4)	(1, 3)
42	6	(8, 3)	(2, 6)
42	6	(7, 0)	(2, 6)
42	6	(9, 3)	(1, 5)
42	6	(15, 3)	(1, 3)
49	7	(11, 6)	(1, 5)
49	7	(18, 5)	(1, 3)
49	7	(7, 7)	(0, 7)
49	7	(8, 5)	(3, 8)
56	8	(10, 2)	(2, 6)
56	8	(11, 5)	(2, 6)
56	8	(12, 4)	(1, 5)
56	8	(20, 4)	(1, 3)
56	8	(8, 5)	(0, 7)

TABLE I: The clusters that have a state at filling $f = 1/7$ for $28 \leq N \leq 56$. All the clusters are commensurate to the $\sqrt{7} \times \sqrt{7}$ crystalline structure shown in Fig. 2(b). In all cases there is a unique zero-energy state in 7 different momentum sectors with a leading configuration which resembles Fig. 2(b).

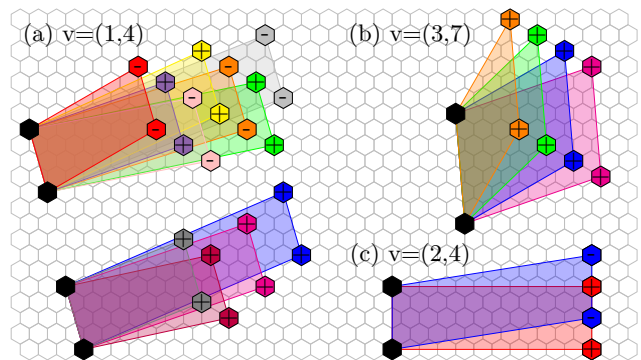


FIG. 4: **A compilation of clusters with zero-energy states at filling $f > \frac{1}{5}$** for (a) $\mathbf{v} = (1, 4)$, (b) $\mathbf{v} = (3, 7)$ and (c) $\mathbf{v} = (2, 4)$. For each case, we mark with different colors the vectors \mathbf{u} of all the clusters with a high filling zero-energy state. The symmetrically equivalent \mathbf{u} 's use the same color (color online). The + and - signs correspond to center of mass momentum $\mathbf{K} = (0, 0)$ and $\mathbf{K} = (0, \pi)$ respectively.

A. Zero-energy states at filling $f \sim 1/7$

The zero energy states near the lower end of the interval $[f_-, f_+]$ have been discussed in Ref. 10 but apparently not in detail. The authors, numerically, identified only one cluster, $(7, 0) \times (0, 7)$, having a zero energy state at $f = 1/7$. In contrast, we found striking behaviors exactly at $f = 1/7$ for a variety of clusters.

In Table I we present all the clusters with zero-energy states at $f = 1/7$. Interestingly, those are all of the distinct clusters which are commensurate with the $\sqrt{7} \times \sqrt{7}$ crystalline order shown in Fig. 2(b), i.e. both the vectors \mathbf{u} and \mathbf{v} are lattice vectors of that pattern. The leading configurations of that wave function consist of the seven possible translations of the same perfectly ordered pattern.

We find that those zero-energy states are gapped.

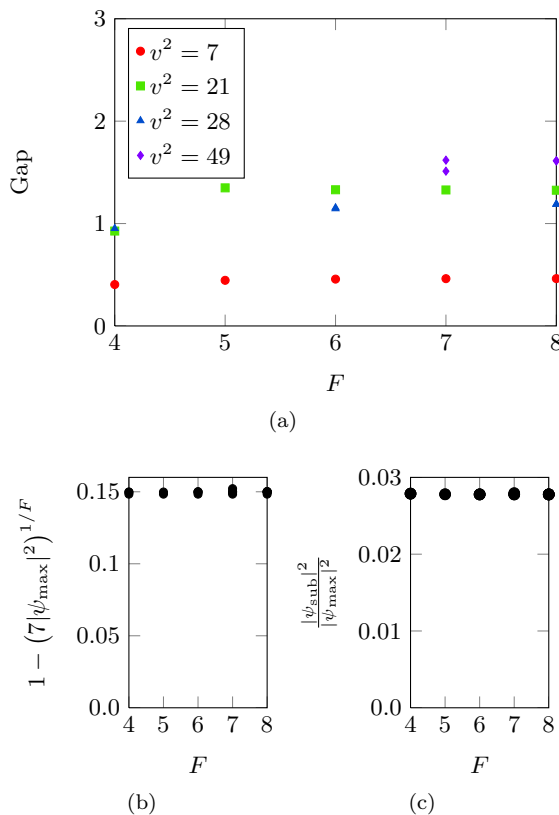


FIG. 5: Nature of the zero-energy states listed in Table I. (a) The gap of the unique zero-energy state at $f = 1/7$ as a function of the number of particles F . The 3-leg ladders with $v = (1, 3)$ approach a different value for large F than the rest of the clusters. The geometry of the ladders is such that each unit cell of the $\sqrt{7} \times \sqrt{7}$ crystalline order wraps around it self. (b) The quantity $q = 1 - (7|\psi_{\max}|^2)^{1/F}$, the inferred probability per crystal site to find its fermion one site away, is independent of system size, as expected for long-ranged crystalline order. (c) The weight of a sub-leading configuration (in which one fermion is off its site) relative to a leading one. This is also independent of F and approximately equal to $\frac{1}{6}q/(1 - q)$, as expected.

Shown in Fig. 5(a) is the gap plotted against the cluster size, grouped as strips of different widths. We notice that the strip of shortest width, $\mathbf{v} = (1, 3)$ [i.e. three-leg ladders], seems to have a gap approaching 0.46, whereas the strips with longer \mathbf{v} have gaps larger than 1. We remark that the three-leg ladders are not representative of the thermodynamic limit, since the nearest neighbors transverse to the strip are images of the same fermion.

It is possible to understand the crystalline order within a very simple picture, based on an eigenstate in which the symmetry is already broken. Our main hypothesis is that, to a good approximation, the probability $|\psi|^2$, as a function of configuration, is a direct product of independent wavefunctions of each fermion hopping in a separate hexagon of 7 sites, centered on one of the ideal crystal sites. Within each hexagon the weight is $(1 - q)$ on the

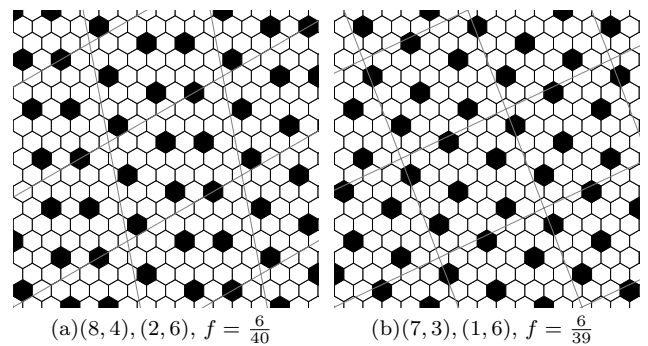


FIG. 6: The leading configurations for two particular clusters in the interval $1/7 < f < 0.156$. Both correspond to unique and gapped zero energy states at a particular momentum sector. Both of them have the form of stripes of the $f = 1/7$ crystal separated by domain walls.

center site and $q/6$ on the other sites. The leading configuration is when every fermion is on a center site and the leading weight is $|\psi_{\max}|^2 = (1 - q)^F/7$, where the factor of 7 signifies that the actual eigenstate in any momentum sector is a superposition of seven shifted copies of that eigenstate, which have no configurations in common. Indeed in Fig. 5(b) we plot $q = 1 - (7|\psi_{\max}|^2)^{1/F}$ and we find it independent of F with a limiting value of $q \approx 0.148$.

A sub-leading term in the many-fermion wavefunction (as we verified by inspecting the wavefunction) goes with a configuration in which each of $(F - 1)$ fermions is on the center site of its hexagon, but one fermion is on a neighbor site in its hexagon. Hence the sub-leading amplitude should be $|\psi_{\text{sub}}|^2 = \frac{1}{7} \frac{q}{6} (1 - q)^{F-1}$ and $|\psi_{\text{sub}}|^2 / |\psi_{\max}|^2 = q/6(1 - q)$. According to Fig. 5(c), the limiting value of this ratio is 0.028 which corresponds to $q \approx 0.144$. The good agreement of this to the q value mentioned above supports the validity of our simple picture.

The character of the $f = 1/7$ zero energy ground states survives in a small interval, $1/7 < f < 0.156$ in which only some of the clusters have zero energy states, in only a few momentum sectors. Similarly to the $f = 1/7$ case, those states are found to be unique and protected by a gap which gets smaller as the f of these states gets larger. The inset of Fig. 1 shows that the energy vs filling does exhibit a gap in this interval. Looking at the structure of the wave function reveals that the leading configuration (and the symmetrically equivalent ones) constitute an $f = 1/7$ pattern with a domain wall as shown in Fig. 6 whereas the subleading configurations are those with one fermion hopped out of this pattern.

B. Zero-energy states at filling $f \gtrsim 1/5$

The existence of zero-energy states for filling $f > 1/5$ has been a subject of debate. Jonsson in a recent study¹¹

N	F	f	\mathbf{u}	\mathbf{v}	\mathbf{K}	N_{lead}	W_{lead}	ΔE
28	6	0.2143	(8, 4)	(1, 4)	(0, π)	9	0.6667	1.1038
33	7	0.2121	(9, 3)	(1, 4)	(0, 0)	12	0.4893	0.9327
38	8	0.2105	(10, 2)	(1, 4)	(0, π)	16	0.3590	0.8290
43	9	0.2093	(12, 5)	(1, 4)	(0, 0)	20	0.2475	0.7068
48	10	0.2083	(13, 4)	(1, 4)	(0, π)	25	0.1705	0.5978
53	11	0.2075	(14, 3)	(1, 4)	(0, 0)	30	0.1127	0.5075
58	12	0.2069	(16, 6)	(1, 4)	(0, π)	36	0.0745	0.4364
29	6	0.2069	(8, 3)	(1, 4)	(0, 0)	2	0.3252	0.2407
34	7	0.2059	(9, 2)	(1, 4)	(0, 0)	16	0.4028	0.6079
44	9	0.2093	(12, 4)	(1, 4)	(0, 0)	40	0.2127	0.3289
54	11	0.2037	(15, 6)	(1, 4)	(0, 0)	80	0.1001	0.3097
44	9	0.2093	(11, 0)	(2, 4)	(0, 0)	8	0.1337	0.7345
44	9	0.2093	(12, 2)	(2, 4)	(0, π)	8	0.1337	0.6338
24	5	0.2083	(6, 6)	(3, 7)	(0, 0)	4	0.6379	0.9461
34	7	0.2059	(7, 5)	(3, 7)	(0, 0)	14	0.6198	0.6386
44	9	0.2093	(8, 4)	(3, 7)	(0, 0)	32	0.4144	0.5823
54	11	0.2037	(9, 3)	(3, 7)	(0, 0)	66	0.2531	0.7621

TABLE II: Clusters in which a high filling zero-energy state is found. N is the number of sites, F the number of fermions, and $f = F/N$ is the filling; the vectors \mathbf{u} and \mathbf{v} define the cluster unit cell. In each case, there is a unique zero-energy eigenfunction in momentum sector \mathbf{K} . N_{lead} is the number of terms in that (normalized) wavefunction having identical maximum amplitude $|\psi_{\text{lead}}|$ and not equivalent by translations; $W_{\text{lead}} = N_{\text{lead}}|\psi_{\text{lead}}|^2$ is the combined weight of such terms; it can be seen that W_{lead} tends to vanish with increasing system size. Also, ΔE is the gap to the next state in the same sector (of F and \mathbf{K})

analyzed a collection of homology cycles on the triangular grid and showed that for a specific class of configurations there is an upper bound on the existence of zero-energy ground states which is $f = 1/5$. Moreover, Huijse *et al.*¹⁰ through extensive exact diagonalization studies and analytical arguments found agreement of the Jonsson conjecture with numerical results on the triangular grid. Our studies have found zero-energy states beyond the filling $f = 1/5$.

1. Occurrence and excitation gaps

Figure 4 displays all the clusters where $f > 1/5$ zero-energy states are observed. Such states are observed in both elongated and roughly isotropic cluster shapes, suggesting that their existence is not an artifact of the aspect ratio (but it could be finite-size related). We find that for a few clusters there are unique zero-energy states at a single momentum sector which is either $(0, 0)$ or $(0, \pi)$ depending on the cluster. Table II shows all the clusters and momentum sectors with $f > 1/5$ zero-energy states.

The shapes of cluster that tend to support zero-energy states with $f \gtrsim 1/5$ are shown in Table II. As illustrated

in Figure 4, these clusters fall into sequences, each having fillings that approach $f = 1/5$ from above in the thermodynamic limit.

The clusters with $\mathbf{v} = (1, 4)$ in the beginning rows of Table II fall in a sequence with $\mathbf{u} = (0, 2 - \epsilon) + m(1, -1)$ modulo \mathbf{v} for $\epsilon = 0, 1$ and $m = 6, 7, \dots, 12$. These have $F = m$ fermions in $N = 5m + \epsilon - 2$ sites so the filling is $f = \lceil [5 + (\epsilon - 2)/m] \rceil$. The momentum where the zero-energy states appear is $\mathbf{K} = (0, 0)$, except when $\epsilon = 0$ and m is even we have $\mathbf{K} = (0, \pi)$. (The $\epsilon = 1$ zero-energy states appear only for $m = 6, 7, 9, 11$, so they may be a finite size effect.) Also, the last rows of the table are four clusters having $\mathbf{v} = (3, 7)$ and the other vector in a sequence $\mathbf{u} = 4(1, 2) + m(1, -1)$ with $m = 2, \dots$ (the $m = 1$ cluster was omitted from the table because $N < 25$.) These have $F = 2m + 1$ fermions in $N = 10m - 1$ sites, for a filling which is obviously greater than $1/5$.

The gaps of the $\mathbf{v} = (1, 4)$ ladders shown in table II can be fitted to a decaying exponential function of the system size, which extrapolates to a gapless state in the thermodynamic limit. Indeed, within this series the gaps fit well to $\exp(-\text{Const}N^2)$. We know (see Section IV A) that in a crystal the amplitudes should decay exponentially, so this is suggestive of a liquid phase, or perhaps a crystal with unbounded fluctuations, similar to a classical crystal in $d = 2$ at nonzero temperature.

The clusters at width $\mathbf{v} = (3, 7)$, the only ones in this table that are plausibly two-dimensional, have the largest gaps – of order unity – suggesting the presence of a gapped state in this filling. For the $\mathbf{v} = (2, 4)$ ladders the data are insufficient to draw a conclusion.

2. Density correlations and coincident wavefunction amplitudes

To further understand the ground state wavefunctions at fillings $f > 1/5$, we examined the density correlation functions in these wavefunctions [Fig. 7(a,c,e)]. The density-density correlation function in each of these states suggests a non-trivial order, which in some cases it is stripe-like, whereas in others it corresponds to a more isotropic structure [cf. Fig. 7 (a,c,e).]

We also examined the leading amplitude configurations in each wavefunction [examples in Fig. 7(b,d,f)]. The combined weight of those leading configurations appears to vanish with the system size, but that happens even when the leading configurations strongly dominate the behavior, e.g. in the $f = 1/7$ crystal [see Figure 5(b)] where they decay as $\exp(-\text{const}N)$. For the $f \gtrsim 1/5$ zero-energy wavefunctions, the decay appears to be faster, perhaps as $\exp(-\text{const}N^2)$. (This can be inferred from the last column of Table II.) This indicates that the $f \gtrsim 1/5$ states are less ordered than the $f = 1/7$ crystal.

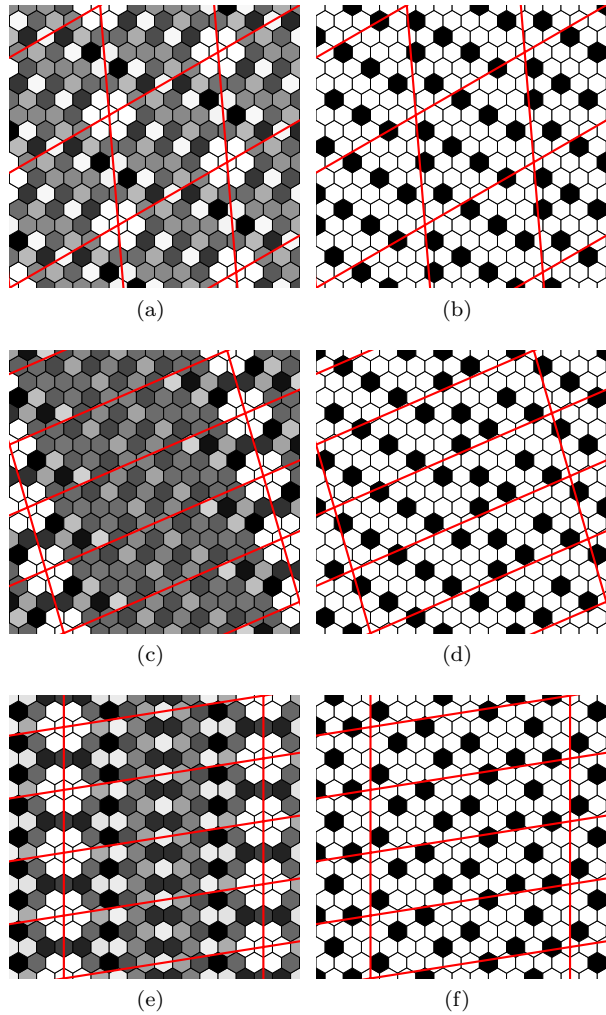


FIG. 7: The density density correlation (left panels) and the leading configuration (right panels) for the unique zero-energy states at $f \gtrsim 1/5$. In each case, the unit cells are marked by light dotted lines, starting from the left lower corner. note in (c,d) the second set of (1,4) edges does not pass through the panel, while in (e,f) the second set of (2,4) edges is the panel's left boundary. (a), (b) (8,4), (3,7) at $f = 9/44$, (c), (d) (15,6), (1,4) at $f = 11/54$, (e), (f) (12,2), (2,4) at $f = 9/44$. (a), (c) and (d): map of the density density correlation $\langle \hat{\rho}(\mathbf{r})\hat{\rho}(\mathbf{r} + \mathbf{R}) \rangle$ as a function R . In each panel many copies of the unit cell (marked by thin lines) are shown; darker grays indicate higher correlations, except that the origin is made white for better contrast. At certain separations the correlation function is almost zero, but its only exact zeroes are the nearest neighbors to the origin (and points equivalent by translation), which are excluded by the reference fermion. (b), (d) and (f): one of the leading configurations all having identical weights as shown in table. II. Each configuration has the covering property, i.e. every unoccupied site is neighbor to at least one fermion, and consequently the potential energy has a the minimum value.

3. “Magic” wavefunctions

All of the $f \gtrsim 1/5$ zero-energy wavefunctions have the unusual property of “coincident amplitudes”, meaning there are multiple configurations having the identical amplitudes (apart from sign). We have checked this only for those of maximum amplitude, i.e. the “leading” configurations. In all cases we know, these leading configurations all have the minimum possible potential energy V_{\min} .

The coincident-amplitude property is somewhat reminiscent of the Rokhsar-Kivelson (RK) eigenstates²², in which *all* configurations have equal weight. Generalizations of the RK construction^{23–26} might inspire future possibilities of analytic approaches to the supersymmetric model: e.g. one might construct the *exact* ground state wavefunction for the given cluster; or possibly, the quantum ground state might have the same weights as the Boltzmann ensemble of some classical model, which would assist in understanding the degree and nature of order in the thermodynamic limit.

In the present work, we could only recognize the coincident-amplitude property when the zero state is unique/non-degenerate in its momentum sector; otherwise, the condition is ill-defined, since the amplitudes depend on which linear combination of the degenerate states is provided by the Lanczos solver. Thus, such wavefunctions are likely to be identified only near the endpoints of the interval of fillings with zero energy states, i.e. around $f \approx 0.15$ or $f \approx 0.20$; in practice we only identified them for $f \gtrsim 0.20$.

Furthermore, the leading configurations in Fig. 7(b) and (d) are dominated by fragments of the $f = 1/5$ crystal from Fig. 2(e), which is the densest simple crystal in which the fermions have available hoppings and the potential energy has the minimum value of F . In the actual leading configurations, some triangles from the other crystals in Fig. 2 get mixed in, which allows more quantum fluctuations and lowers the kinetic energy. Notice these are very anisotropic structures, in that almost all allowed hoppings are in the same directions (aligned with the long direction of the $f = 1/5$ triangles). The configuration in Fig. 7(f) appears to be dominated instead by rows of the $f = 1/6$ cell from Fig. 2(d) which explains why the (12,2),(2,4) cluster is more isotropic than the other clusters.

V. CONCLUDING REMARKS

The exact diagonalization study we performed on a variety of finite clusters illuminated certain aspects of the phase diagram of the supersymmetric lattice model of Eq. 1. The main findings are the presence of gapped crystalline zero energy states for $f \sim 1/7$, gapless quasi-ordered states for $f > 1/5$ in violation of Jonsson’s conjecture and a zero state entropy which peaks at $f \sim 0.178$. Our results are not conclusive about the exact phase diagram in several places and we now speculate on

the possible scenarios, in light of our results.

Our finding of an energy gap at $f = 1/7$, as shown in Section IV A, implies that the Fermi liquid phase cannot connect continuously to the zero state at $f = 1/7$. Either there is a first-order transition, so that coexistence of the Fermi liquid with the $f = 1/7$ crystal is present in a small interval below $f = 1/7$, or else the crystal phase extends (with $E > 0$) to lower fillings. A specific mechanism for the latter alternative is that the crystal is doped with fermion vacancies (which thus form a Fermi liquid within the crystal), which destabilizes (i.e. melts) the crystal at a sufficient density of vacancies. Additional studies will be needed to decide between these scenarios.

The next open question is the behavior immediately above $f = 1/7$. Our results suggest there is a family of states made by combining strips of the $f = 1/7$ and $f = 1/6$ crystals: each such state is a zero-energy state similar to the $f = 1/7$ crystal in being gapped and in having a crystal-like symmetry breaking. This would imply a dense set of rational fillings above $f = 1/7$ that are, in some sense, isolated from each other rather than forming a continuum.

We conjecture that the fillings in the interval $0.156 < f < 0.2$ are all *gapless* in the thermodynamic limit, based on the numerical observations discussed at the end of section III B. If this is true in the thermodynamic limit, it would fit naturally with a liquid-like phase, in which the wavefunction has a vast number of important configurations, each of them mixing the units from all the crystals shown in Figure 2, and the important configurations are accessible to each other by small steps corresponding to local rearrangements of these units. This scenario could be reconciled with the *apparent* increase of the gaps away from the center of that interval at $f = 0.178$, in the following speculative picture. Say that the entropy (per site) of *low-energy* states has a similar dependence on f as the entropy of *zero-energy* states. Then, for f near the endpoints and with a finite size N , the expected number of low-energy states in a cluster might be less than unity, meaning that sometimes the cluster has none.

For the clusters with zero energy states at $f \gtrsim 1/5$ we find non-degenerate zero-energy states in just a single momentum sector. The filling of those states seems to converge to $1/5$ for increasing system size; furthermore, we checked that if we make a new cluster that simply doubles one of these clusters, and keep the same f , the larger cluster does *not* generically have a zero energy state. It is unclear how our results relate to the thermodynamic limit. These states showed more than one kind of ordering tendency, depending on the cluster (as noted at end of Sec. IV B), but the tendency is that most of the available hoppings lie along a particular axis. This might be explained by a local order incorporating many fragments of the $f = 1/5$ crystal [Figure 2(e)], which only allows hops along one lattice axis.

The corresponding wave functions at $f \gtrsim 0.2$ are dominated by many configurations with equal (“coincident”) amplitudes that always belong to the lowest potential energy sector, (Sec. IV B 3). We certainly cannot rule out the possibility that similar wavefunctions exist for other zero states, since we are only able to detect this property when there is a unique zero state in a particular sector. The regularities in these wavefunctions suggest the possibility that some analytic structure, even an exact solution, may be found for the zero-energy ground states.

Finally, at both ends of the range $(1/5, 1/3)$, we naively expect some form of coexistence of a hole-rich phase with the $f = 1/3$ inert crystal, or this crystal doped by an array of stripe-walls. The $E(f)$ curve shows a straight line at very nearly the slope expected from a stripe-wall array. However, the picture from our exact diagonalizations is not quite consistent with any scenario: the $E(f)$ curve due to an array ought to curve upwards rather than be a straight line; in contrast, in the case of coexistence a straight tie-line is not expected in a finite system due to the additional cost of the domain wall. One intriguing possibility is that the domain-wall cost is zero: that is, the phase at $f \gtrsim 0.5$ and the stripe-wall array dissolve into each other. This is not implausible: the stripe-wall array is an $f = 1/3$ crystal with some units of the $f = 1/5$ crystal appearing along the stripe-walls, and it has hoppings entirely along one of the crystal axis directions. Thus, its properties are similar to some of the zero states we observed at $f \gtrsim 1/5$, so perhaps the quasi-order we found in those states is simply a generalization of the stripe-wall kind of state. It is worth noting that in Ref. 10, it was argued that the ground state at $f = 1/4$ is gapped. It would be interesting to identify the deep origin of this gap, given that stripe-walls, as well as coexisting stripe-walls should be by all odds gapless.

Overall, a possible interpretation of our consistent finding of a strong bias towards ordering in finite clusters, is that the supersymmetric model is a multicritical point in the parameter space of Hamiltonians, so that many different kinds of order may be stabilized by infinitesimal perturbations. The various sizes and shapes of the system introduce similar biases, so that different large clusters may be occupied by fragments of competing ordered states which are all valid and degenerate in the thermodynamic limit. In future studies, it may be worth to add terms to the Hamiltonian (1) so as to explore the possible relation of supersymmetry to multicriticality^{6,27}.

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 - ²⁸ Such results are based on cohomology theory (Refs. 3 and 7). The study in Ref. 11 focused only on a certain kind of homology cycles, the so-called “cross-cycles”, where each such cycle coincides with the fundamental cycle of the boundary complex of a cross-polytope. In this case, the homology of the independence complex on the triangular lattice appears tractable, even though it is not exhaustive. The count of zero-energy states is better understood in the square lattice case, where it was even shown that a 1-1 correspondence exists between the possible zero-energy states and the tilings of the 2D hard-squares model at negative fugacity $z = -1$.
 - ²⁹ In Refs. 15 and 14, they were called “stripes”.